EQUILIBRIUM POINTS OF LOGARITHMIC POTENTIALS ON CONVEX DOMAINS

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ABSTRACT. Let D be a convex domain in $\mathbb{C}$. Let $a_k > 0$ be summable constants and let $z_k \in D$. If the z_k converge sufficiently rapidly to $\eta \in \partial D$ from within an appropriate Stolz angle then the function $\sum_{k=1}^{\infty} a_k/(z - z_k)$ has infinitely many zeros in D. An example shows that the hypotheses on the z_k are not redundant, and that two recently advanced conjectures are false.
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1. Introduction

A number of recent papers [4, 5, 9, 10] have concerned zeros of functions

$$f(z) = \sum_{k=1}^{\infty} \frac{a_k}{z - z_k}, \tag{1}$$

and in particular the following conjecture [4].

Conjecture 1.1 ([4]). *Let f be given by (1), where $a_k > 0$ and*

$$z_k \in \mathbb{C}, \quad \lim_{k\to\infty} z_k = \infty, \quad \sum_{z_k \neq 0} \left| \frac{a_k}{z_k} \right| < \infty. \tag{2}$$

Then f has infinitely many zeros in $\mathbb{C}$.

The assumptions of Conjecture 1.1 imply that f is meromorphic in the plane and, assuming that all z_k are non-zero, $f(z)$ is the complex conjugate of the gradient of the associated subharmonic potential $u(z) = \sum_{k=1}^{\infty} a_k \log|1 - z/z_k|$. Moreover, Conjecture 1.1 has a physical interpretation in terms of the existence of equilibrium points of the electrostatic field arising from a system of infinite wires, each carrying a charge density a_k and perpendicular to the complex plane at z_k [8, p.10]. Conjecture 1.1 is known to be true when $\sum_{|z_k|\leq r} a_k = o(\sqrt{r})$ as $r \to \infty$ [4, Theorem 2.10] (see also [6, p.327]), and when $\inf\{a_k\} > 0$ [5] (see also [9]).

An analogue of Conjecture 1.1 for a disc was advanced in [3, Conjecture 2].

Conjecture 1.2 ([3]). *Let $0 < \rho < \infty$ and $\theta \in \mathbb{R}$. Let f be given by (1), where*

$$z_k \in \mathbb{C}, \quad |z_k| < \rho, \quad \lim_{k\to\infty} z_k = \rho e^{i\theta}, \quad a_k > 0, \quad \sum_{k=1}^{\infty} a_k < \infty. \tag{3}$$

Then f has infinitely many zeros in $|z| < \rho$.

If f satisfies the assumptions of Conjecture 1.2 then $\overline{f} = \nabla u$ in $|z| < \rho$, where $u(z) = \sum_{k=1}^{\infty} a_k \log|z - z_k|$. Obviously there is no loss of generality in assuming

that $\rho = 1$ and $\theta = 0$ in Conjecture 1.2. Writing

$$w = \frac{1}{1-z}, \quad w_k = \frac{1}{1-z_k}, \quad f(z) = wF(w), \tag{4}$$

where

$$F(w) = \sum_{k=1}^{\infty} \frac{a_k w_k}{w - w_k}, \tag{5}$$

it is easy to verify that Conjecture 1.2 is equivalent to the following.

Conjecture 1.3. *Let F be given by (5), where*

$$w_k \in \mathbb{C}, \quad \mathrm{Re}\, w_k > \frac{1}{2}, \qquad \lim_{k\to\infty} w_k = \infty, \quad a_k > 0, \quad \sum_{k=1}^{\infty} a_k < \infty. \tag{6}$$

Then F has infinitely many zeros in $\mathrm{Re}\, w > 1/2$.

With the assumptions (6), the function $F(w)$ in (5) is evidently meromorphic in the plane. In §2 an example satisfying (5) and (6) will be constructed, such that $F(w)$ has no zeros in $\mathbb{C}$. Thus Conjectures 1.2 and 1.3 are false, and there is no direct analogue of Conjecture 1.1 for the unit disc.

On the other hand the following theorem shows in particular that if the z_k converge to $\rho e^{i\theta}$ sufficiently rapidly, and if all but finitely many z_k lie in a sufficiently small Stolz angle, then the conclusion of Conjecture 1.2 does hold. It is convenient to state and prove the result when the z_k lie in a convex domain D and the boundary point $\rho e^{i\theta}$ is 1. There then exists (see §4) an open half-plane H such that $D \subseteq H$ and 1 lies on the boundary ∂H, and there is no loss of generality in assuming that H is the half-plane $\mathrm{Re}\, z < 1$.

Theorem 1.1. *Let $D \subseteq \{z \in \mathbb{C} : \mathrm{Re}\, z < 1\}$ be a convex domain such that $1 \in \partial D$. Let f be given by (1), where*

$$z_k \in D, \quad a_k > 0, \quad \sum_{k=1}^{\infty} a_k < \infty. \tag{7}$$

Assume that 1 is a limit point of the set $\{z_k : k \in \mathbb{N}\}$, and that there exist real numbers $\varepsilon > 0$ and $\lambda \geq 0$ such that

$$\sum_{|1-z_k|\leq\varepsilon} |1-z_k|^{\tau} < \infty \quad \textit{for all} \quad \tau > \lambda, \tag{8}$$

and

$$\sup\{|\arg(1-z_k)| : k \in \mathbb{N},\, |1-z_k| \leq \varepsilon\} < C(\lambda) = \frac{\pi}{2\lambda}. \tag{9}$$

Then there exists a sequence (η_j) of zeros of f satisfying $\eta_j \in D, \lim_{j\to\infty} \eta_j = 1$.

Note that (8) implies that $\{z_k : k \in \mathbb{N}\}$ has no limit points z in the punctured disc A given by $0 < |1-z| < \varepsilon$, and that f is meromorphic on A. Moreover, (9) is obviously satisfied if $\lambda < 1$.

2. A counterexample to Conjecture 1.3

Let

$$g(w) = \frac{1}{w(w-2)(e^{w-1}+1)}. \tag{10}$$

Then g has no zeros, but has simple poles at 0, 2 and

$$u_k = 1 + (2k+1)\pi i, \quad k \in \mathbb{Z}. \tag{11}$$

Straightforward computations give

$$\operatorname{Res}(g,0) = \frac{-1}{2(e^{-1}+1)} = -a, \quad \operatorname{Res}(g,2) = \frac{1}{2(e+1)} = b, \tag{12}$$

and, using (11),

$$\operatorname{Res}(g,u_k) = \frac{-1}{u_k(u_k-2)} = \frac{-1}{(u_k-1)^2-1} = \frac{1}{(2k+1)^2\pi^2+1} = c_k. \tag{13}$$

Then b and the c_k evidently satisfy

$$b > 0, \quad c_k > 0, \quad \sum_{k\in\mathbb{Z}} c_k < \infty. \tag{14}$$

Next, let

$$h(w) = -\frac{a}{w} + \frac{b}{w-2} + \sum_{k\in\mathbb{Z}} \frac{c_k}{w-u_k}, \quad L(w) = h(w) - g(w). \tag{15}$$

By (10), (11), (12), (13) and (14) the function $h(w)$ is meromorphic in the plane, and $L(w)$ is an entire function.

Let m be a large positive integer, let $R = 4m\pi$, and use c to denote positive constants independent of m. Then simple estimates give

$$|g(w)| \leq \frac{c}{R^2} \quad \text{for} \quad |w-1| = R \tag{16}$$

and, as $m \to \infty$,

$$|h(w)| \leq \frac{c}{R} + c \sum_{k\in\mathbb{Z},|k|\geq m} c_k + c \sum_{k\in\mathbb{Z},|k|<m} \frac{c_k}{R} = o(1) \quad \text{for} \quad |w-1| = R. \tag{17}$$

Combining (16) and (17) shows that $L(w) \equiv 0$ in (15), so that $h = g$ has no zeros, and applying the residue theorem in conjunction with (16) now gives

$$a = b + \sum_{k\in\mathbb{Z}} c_k. \tag{18}$$

Hence $h(w)$ may be expressed using (18) in the form

$$\begin{aligned} h(w) &= b\left(\frac{1}{w-2} - \frac{1}{w}\right) + \sum_{k\in\mathbb{Z}} c_k \left(\frac{1}{w-u_k} - \frac{1}{w}\right) \\ &= \frac{1}{w}\left(\frac{2b}{w-2} + \sum_{k\in\mathbb{Z}} \frac{c_k u_k}{w-u_k}\right). \end{aligned} \tag{19}$$

By (11), (14) and (19) the function $F(w) = wh(w)$ may be written in the form

$$F(w) = \sum_{k=1}^{\infty} \frac{d_k v_k}{w-v_k}, \quad \operatorname{Re} v_k \geq 1, \quad v_k \to \infty, \quad d_k > 0, \quad \sum_{k=1}^{\infty} d_k < \infty. \tag{20}$$

Here F evidently satisfies the requirements of (5) and (6), but F has no zeros in $\mathbb{C}$, since h has no zeros and $h(0) = \infty$.

Remark. It is conjectured further in [3, Conjecture 6] that if f satisfies (1) and (2) with $a_k \overline{z}_k > 0$ for each k then f has infinitely many zeros in $\mathbb{C}$. The example (20), with $a_k = d_k v_k$ and $a_k \overline{v}_k = d_k |v_k|^2 > 0$, shows that this conjecture is also false.

3. An auxiliary result needed for Theorem 1.1

The proof of Theorem 1.1 rests upon the following proposition, which concerns functions in the plane of the form (5), and uses standard notation from [7, p.42].

Proposition 3.1. *Let $0 < \sigma \leq 1$. Let F be given by (5), where*

$$w_k \in \mathbb{C}, \quad \operatorname{Re} w_k > 0, \qquad a_k > 0, \quad \sum_{k=1}^{\infty} a_k < \infty. \tag{21}$$

Assume that the set $\{w_k : k \in \mathbb{N}\}$ is unbounded and that there exist real numbers $R > 0$ and $\lambda \geq 0$ such that

$$\sum_{|w_k| \geq R} |w_k|^{-\tau} < \infty \quad \text{for all} \quad \tau > \lambda, \tag{22}$$

and

$$s = \sup\{|\arg w_k| : k \in \mathbb{N},\, |w_k| \geq R\} < C(\lambda, \sigma) = \frac{2}{\lambda} \arcsin \sqrt{\frac{\sigma}{2}}. \tag{23}$$

Then there exists a transcendental meromorphic function G with

$$F(w) = G(w)(1 + o(1)) \quad \text{as} \quad w \to \infty, \tag{24}$$

and the Nevanlinna deficiency $\delta(0, G)$ of the zeros of G satisfies $\delta(0, G) < \sigma$. In particular, $F(w)$ has a sequence of zeros tending to infinity.

The zero-free example of (20) has $\lambda = 1$ and $\delta(0, F) = \sigma = 1$, and all its poles lie in $\operatorname{Re} w \geq 1$, so that Proposition 3.1 is essentially sharp.

To prove Proposition 3.1, assume that F is as in the statement of Proposition 3.1. It follows from (22) that the set $\{w_k : k \in \mathbb{N}\}$ has no limit points w with $R < |w| < \infty$. In particular, F is meromorphic in the region $2R \leq |w| < \infty$ with an essential singularity at infinity. The existence of a transcendental meromorphic function G satisfying (24) then follows from a result of Valiron [12, p.15] (see also [2, p.89]). In particular, G is constructed [12] so that F and G have the same poles and zeros in $|w| \geq 2R$. If $|w| \geq 4R$ then (21) gives

$$|F(w)| \leq |F_1(w)| + O(1), \quad F_1(w) = \sum_{|w_k| \geq 2R} \frac{a_k w_k}{w - w_k},$$

so that

$$m(r, G) \leq m(r, F_1) + O(1) = O(1)$$

as $r \to \infty$, by [6, p.327]. Since the poles w_k of G have exponent of convergence at most λ by (22), it follows that G has lower order $\mu \leq \lambda$.

Choose s_0, s_1, s_2 with

$$s < s_0 < s_1 < s_2 < \min\{\pi, C(\lambda, \sigma)\}, \tag{25}$$

where s is as in (23) and satisfies $s \leq \pi/2$ by (21). The proof of Proposition 3.1 requires the following two lemmas.

Lemma 3.1. *The function F satisfies* $\liminf_{r\in\mathbb{R}, r\to+\infty} r|F(-r)| > 0$.

Proof. Let $r > 0$ and write $w_k = u_k + iv_k$ with u_k and v_k real. Let

$$p_k(r) = \mathrm{Re}\left(\frac{w_k}{r + w_k}\right) = \frac{u_k(r + u_k) + v_k^2}{(r + u_k)^2 + v_k^2}.$$

Then (21) gives $p_k(r) > 0$ and there exists $d > 0$ such that $p_1(r) > d/r$ as $r \to \infty$. Hence, again as $r \to \infty$,

$$r|F(-r)| \geq -r\mathrm{Re}\, F(-r) = r\sum_{k=1}^{\infty} a_k p_k(r) \geq r a_1 p_1(r) > a_1 d.$$

□

Lemma 3.2. *There exists $M_1 > 0$ such that $|F(w)| \leq M_1$ for all large w lying outside the region $|\arg w| < s_0$.*

Proof. This follows from (21), (23) and (25), since there exists a positive constant M_2 such that $|w - w_k| \geq M_2|w_k|$ for all such w and all $k \in \mathbb{N}$. □

The proof of Proposition 3.1 may now be completed using Lemmas 3.1 and 3.2. Assume that $\delta(0, G) \geq \sigma$. Then Baernstein's spread theorem [1] gives a sequence $r_m \to \infty$ and, for each m, a subset I_m of the circle $|w| = r_m$, of angular measure at least

$$\min\left\{2\pi, \frac{4}{\mu}\arcsin\sqrt{\frac{\sigma}{2}}\right\} - o(1) \geq \min\{2\pi, 2C(\lambda, \sigma)\} - o(1) \geq 2s_2,$$

using (23) and (25), and such that

$$\lim_{m\to\infty} \frac{\max\{\log|G(w)| : \quad w \in I_m\}}{\log r_m} = -\infty. \tag{26}$$

Let m be large, and consider the function $v(w) = \log|F(w)|$, which is subharmonic on the domain

$$\Omega = \{w \in \mathbb{C} : r_m/4 < |w| < r_m, s_0 < \arg w < 2\pi - s_0\}.$$

Then v is bounded above on Ω, by Lemma 3.2. But the intersection J_m of I_m with the arc $\{w \in \mathbb{C} : |w| = r_m, s_1 < \arg z < 2\pi - s_1\}$ has angular measure at least $2(s_2 - s_1)$, so that standard estimates for the harmonic measure of J_m at $-r_m/2$ now give

$$\omega(-r_m/2, J_m, \Omega) \geq M_3 > 0, \tag{27}$$

where M_3 is independent of m. Since (24) implies that (26) holds with G replaced by F, combining Lemma 3.2 with (27) and the two-constants theorem [11, p.42] leads to

$$r_m F(-r_m/2) \to 0 \quad \text{as} \quad m \to \infty. \tag{28}$$

But (28) contradicts Lemma 3.1, and this completes the proof of Proposition 3.1.

4. Proof of Theorem 1.1

Assume that f and D satisfy the hypotheses of Theorem 1.1. Define F using the transformations (4) and (5). Then F satisfies the hypotheses of Proposition 3.1 with $R = 1/\varepsilon$ and $\sigma = 1$. Thus F has a sequence of zeros tending to infinity, and so f has a sequence (η_j) of zeros with $\lim_{j\to\infty} \eta_j = 1$.

It remains only to show that such a sequence (η_j) exists with, in addition, $\eta_j \in D$, and this is done by a standard argument of Gauss-Lucas type. Let $\eta = \eta_j$ with j large, and assume that $\eta \notin D$. Since D is convex the supremum and infimum of $\arg(z - \eta)$ on D differ by at most π. Hence there exist an open half-plane H, with $D \subseteq H$ and $\eta \in \partial H$, and a linear transformation $u = T(z) = (z - \eta)/a$ mapping H onto $\mathrm{Re}\, u > 0$. Writing $u_k = T(z_k)$ then gives

$$0 = \mathrm{Re}\,(af(\eta)) = -\mathrm{Re}\left(\sum_{k=1}^{\infty} \frac{a_k}{u_k}\right) < 0.$$

This contradiction completes the proof of Theorem 1.1.

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